Enhancement of phase synchronization through asymmetric couplings

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Phase synchronization in lattices of coupled chaotic oscillators is studied. It is found that phase synchronization can be greatly improved by asymmetric biased coupling. The mechanism responsible for this effect is the transition from a localized wave to synchronized flow and nonlocal phase synchronization.

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Phase entrainment among a group of oscillators with distributed natural frequencies is a commonly observed phenomenon in many realistic cases [1,2], for example, in physics (such as coupled lasers, Josephson junction arrays, magnetic resonance, charged wave instabilities in plasma), networks of biological oscillators (electrical synchrony among cardiac pacemakers, heartbeat synchronization with ventilation, resting tremor in Parkinson's disease, flashing of fireflies, chirping of crickets, etc.), and chemical oscillators [3]. In the classical sense, synchronization of periodic self-sustained oscillators is usually defined as locking of the phase, $m\theta_1 - n\theta_2 = \text{const}$ due to weak interaction, while the amplitudes may be quite different. Recently, the notion of synchronization has been extended to chaotic oscillators (driven chaotic oscillators or coupled chaotic oscillators) [4–6]. One of the remarkable findings is the numerical and experimental observations of the phase synchronization (PS) phenomenon in a system of two mutually coupled nonidentical self-sustained chaotic oscillators [7]. The phenomenon is analogous to synchronization of periodic oscillators where only phase locking is important, while their amplitudes remain chaotic and noncorrelated. Clustering PS has been studied in lattices of coupled chaotic oscillators [8,9]. For chaotic systems with broad time scales (strong chaos), it was found that a perfect PS cannot be achieved, while a PS temporally alternating between a number of $m:n$ lockings was observed due to the overlap of $m:n$ Arnold tongues in coupled chaotic oscillators [10].

Since synchronization is of great importance in practice, a crucial topic is how to optimize the synchronization among elements. Phase is a degree that can be relatively easy to tame in chaotic motions; therefore, it is desirable to investigate the optimization of PS. In the present paper we study the effect of a biased coupling on phase entrainment of chaotic oscillators. We reveal that PS can be greatly enhanced with increase of the biased coupling [11]. We attribute this enhancement to the delocalization transition of a localized synchronized wave and its consequent nonlocal PS.

The model we adopt to study the PS behavior is a lattice of nearest-neighbor-coupled chaotic oscillators:

$$\dot{\mathbf{X}}_i = \mathbf{F}(\mathbf{P},\mathbf{X}_i) + (K+\delta)\mathbf{D}(\mathbf{X}_{i+1} - \mathbf{X}_i) - (K-\delta)\mathbf{D}(\mathbf{X}_i - \mathbf{X}_{i-1}), \tag{1}$$

where $\mathbf{X}_i(t)$ represents the flow of the ith oscillator and $\mathbf{F}$ is a nonlinear function with $\mathbf{P}$ the parameter set. Due to the nonlinearity, the flow of individual oscillators in phase space is usually chaotic. Two types of coupling, i.e., diffusive coupling with strength K and a drift (gradient) coupling δ, are introduced, and $\mathbf{D}$ denotes a coupling matrix. The total coupling is asymmetric because of the presence of the drift coupling. In this paper we study the case of a three-dimensional chaotic flow $\mathbf{X}_i = (x_i, y_i, z_i)$ and x coupling, i.e., $\mathbf{D}_{11} = 1$ and $\mathbf{D}_{ij} = 0$ for $i \neq 1$ and $j \neq 1$. A paradigmatic example in studies of chaos is the Rössler oscillator. When oscillators are coupled in the above way, the equation of motion can be written as

$$\begin{aligned} \dot{x}_i &= -\omega_i x_i - z_i + (K+\delta)(x_{i+1} - x_i) - (K-\delta)(x_i - x_{i-1}), \\ \dot{y}_i &= \omega_i x_i + a y_i, \\ \dot{z}_i &= f + z_i(x_i - c), \end{aligned} \tag{2}$$

where $a = 0.165$, $f = 0.2$, and $c = 10$ are applied in this paper. The parameters $\omega_i = 1.0 + \Delta_i$, where the misfit $\Delta_i \in [-\Delta, \Delta]$ and obeys a uniform distribution. Although the definition of the phase for a general chaotic oscillator is still a challenging topic because of the multiple rotation centers [12], the phase of the Rössler system can be conveniently introduced as

$$\theta_i(t) = \tan^{-1}[y_i(t)/x_i(t)]. \tag{3}$$

Furthermore, as oscillators are coupled to each other, PS can be achieved when the 1:1 locking condition $|\theta_i - \theta_j| < \text{const}$ or $\Omega_i = \Omega_j$ $(i \neq j)$ is satisfied, where $\Omega_i = \lim_{T\to\infty}(1/T)\int_0^T \dot{\theta}_i(t)\,dt$ [13]. In Fig. 1, for diffusive coupling ($\delta = 0$), we show the PS cascade tree Ω_i versus K for different numbers of oscillators. These bifurcation trees exhibit a typical cascade from partial PS to full PS, which are very similar to the phase entrainment behavior for coupled periodic oscillators [14]. The tree structure reveals an intrinsic order embedded in high-dimensional chaotic motions. The route from partial to full PS has also been discussed by Osipov *et al.* [8] in terms of the Ω_i vs i profile.

Now we focus on the effect of nonzero drift coupling, i.e., the coupling among nearest neighbors is asymmetric. In Fig.

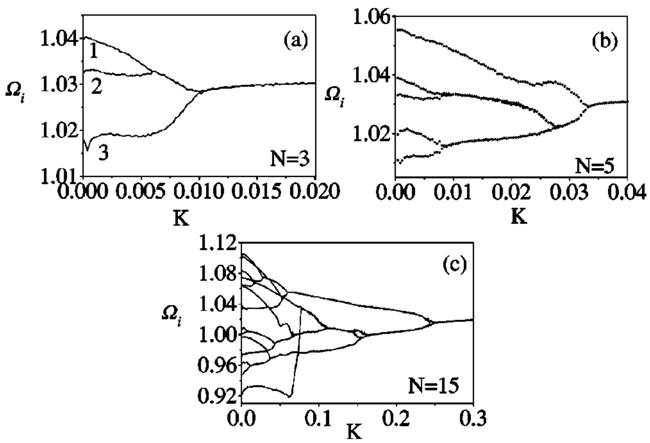

FIG. 1. The bifurcation trees Ω_i against the diffusive coupling strength for (a) $N=3$, (b) $N=5$, and (c) $N=15$.

2, the Ω_i vs i profiles are plotted for $N=100$, $K=0.6$, $\delta=0.0$ in 2(a), and $\delta=0.3$ in 2(b). In Fig. 2(a), the system lies in the partial PS state, i.e., a number of plateaus correspond to synchronized clusters satisfying $\Omega_i=\Omega_j$ $(i\neq j)$. As the drift coupling is applied, a full PS can be achieved, i.e., all oscillators are phase locked to each other. This is shown in Fig. 2(b), where several clusters merge into a single plateau. Therefore, the system can reach the full PS under the breaking of coupling symmetry. In Fig. 2(c), we plot the profile $A_i^{\max}$ for $\delta=0$, where $A_i^{\max}=\max[\sqrt{x_i^2(t)+y_i^2(t)}]$ as $t\to\infty$. Interestingly, we find that, although oscillators may become phase synchronized, their amplitudes may be quite different. This typical breather synchronization is a consequence of oscillation quenching, i.e., the suppression of oscillations. The quenching effect can be eliminated by applying a drift coupling. In Fig. 2(d), the profile of $A_i^{\max}$ corresponding to Fig. 2(b) is given, where we find all oscillators now execute oscillations with almost the same amplitude.

Enhancement of PS via asymmetric coupling can be better shown by measuring a quantity that signifies the degree of PS. We study the case of 100 coupled Rössler oscillators. In

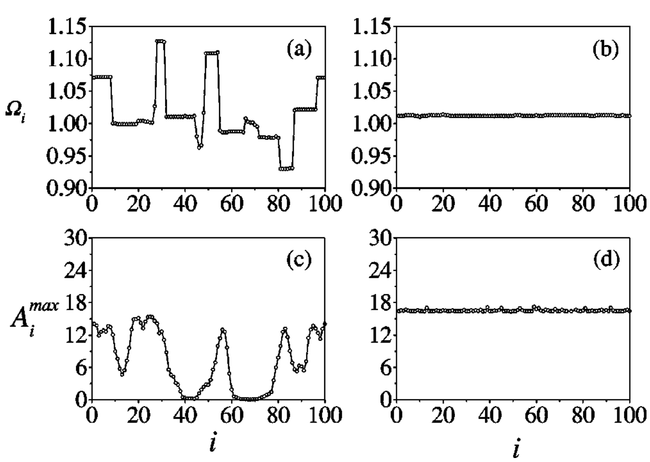

FIG. 2. (a) The phase synchronization profile Ω_i versus i for $K=0.6$ and $\delta=0.0$; (b) the same as (a) with $\delta=0.3$; (c) the profile of maximum oscillation amplitude with parameters the same as (a); (d) the profile of maximum oscillation amplitude with parameters the same as (b).

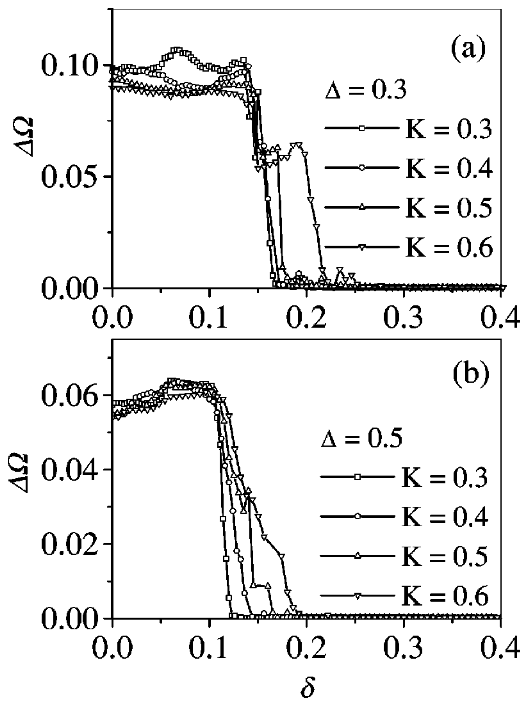

FIG. 3. The difference $\Delta\Omega$ varying with the coupling bias δ for different diffusive coupling strengths. (a) $\Delta=0.3$, (b) $\Delta=0.5$.

Fig. 3(a), for the case $\Delta=0.3$, we measure the difference among average winding numbers by introducing

$$\Delta\Omega=\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\Omega_i-\overline{\Omega})^2}, \tag{4}$$

where $\overline{\Omega}=(1/N)\Sigma_{i=1}^{N}\Omega_i$. It is found that for $K=0.3,0.4,0.5$, and 0.6, $\Delta\Omega$ drops sharply to zero at about $\delta\approx0.16,0.20,0.19$, and 0.25, respectively, signifying a transition from partial PS to complete PS. Figure 3(b) gives the behavior of the transition for $\Delta=0.5$. A similar transition to the full PS state can be identified, where for $K=0.3,0.4,0.5$, and 0.6 $\Delta\Omega$ becomes zero at approximately $\delta=0.123,0.144,0.165$, and 0.186, respectively.

To reveal the mechanism for this transition, we study the case of the PS dynamics of a lattice of coupled *identical* Rössler oscillators responding to a periodic driving signal:

$$\begin{aligned}
\dot{x}_1&=-\omega x_1-z_1+(K+\delta)(x_2-x_1)-K[x_1-10\sin(\omega_0 t)],\\
\dot{y}_1&=\omega x_1+ay_1,\\
\dot{z}_1&=f+z_1(x_1-c),\\
\dot{x}_i&=-\omega x_i-z_i+(K+\delta)(x_{i+1}-x_i)-(K-\delta)(x_i-x_{i-1}),\\
\dot{y}_i&=\omega x_i+ay_i,\\
\dot{z}_i&=f+z_i(x_i-c)\quad\text{for}\quad i\neq1.
\end{aligned} \tag{5}$$

Here $\omega=1.2$, $\omega_0=1.0$, $K=0.9$, and a,f,c are the same as before. When there is no bias for the coupling, i.e., $\delta=0$, one may see from the profile of Ω_i vs i in Fig. 4(a) that only the site adjacent to the periodic driving can be synchronized, that

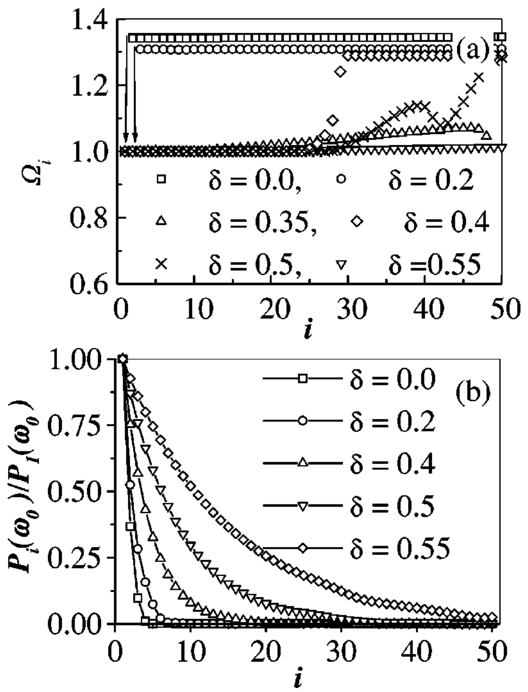

FIG. 4. (a) The PS profile Ω_i versus i for 50 coupled identical oscillators ($\omega = 1.2$) with the first one driven by a harmonic wave $10\sin(t)$; the transition from partial PS to full PS can be observed. (b) The power of the ω_0 component for each oscillator $P_i(\omega_0)/P_1(\omega_0)$ for different biases of the coupling.

is, the harmonic synchronized wave is localized and can propagate only to its neighbor oscillator. As the bias δ is applied, we find that more and more oscillators are phase entrained by the periodic signal. This signifies the delocalization of the harmonic wave. At $\delta_c \approx 0.55$, all oscillators are synchronized to the harmonic wave. Therefore the synchronized wave can propagate along the whole array, which brings all oscillators to the PS state. In Fig. 4(b), we give the Fourier analysis of the evolution of each oscillator, where the power of the ω_0 component for each oscillator,

$$P_i(\omega_0) = \left| \int_{-\infty}^{\infty} x_i(t) \exp(i\omega_0 t) dt \right|, \tag{6}$$

is shown for different biases of the coupling. We use the normalized power $P_i(\omega_0)/P_1(\omega_0)$ to get a unified description. For $\delta = 0$, we find that for $i > 4$ $P_i(\omega_0)/P_1(\omega_0) \rightarrow 0$ as i increases, i.e., the synchronized wave is propagated within a short range. With increasing bias, the synchronized wave becomes less localized and can be transported to a site far from the first one, as shown for $\delta = 0.2, 0.4$, and 0.5. At $\delta \approx 0.55$, the synchronized wave can propagate along the whole array. The delocalization transition of the synchronized wave causes the oscillators far from the driving signal to be phase locked to their neighbors. If the synchronized wave is chaotic, i.e., if we apply a chaotic oscillator instead of a periodic oscillator as our driving signal, a similar conclusion is reached. Therefore, the enhancement of PS through asymmetric coupling originates from the delocalization transition of the synchronized wave.

When one applies a harmonic wave to drive a chain of nonidentical oscillators, i.e., where all oscillators perform oscillations with different intrinsic frequencies, the propagation of a synchronized wave can cause oscillators whose intrinsic frequencies are near to that of the harmonic wave to become phase synchronized (but these oscillators may be nonadjacent on the lattice), even if the harmonic component ω_0 is very weak. This forms nonlocal PS clusters. Nonlocal PS satisfies the condition $\Omega_i = \Omega_j$, where i, j are not neighbors, $i \neq j+1$ or $j-1$ [15]. These clusters in turn drive their nearby oscillators into the PS state, and thus larger clusters may form. Therefore nonlocal PS plays a crucial role in forming larger solid PS clusters. A typical instance is the coupled Lorenz oscillators

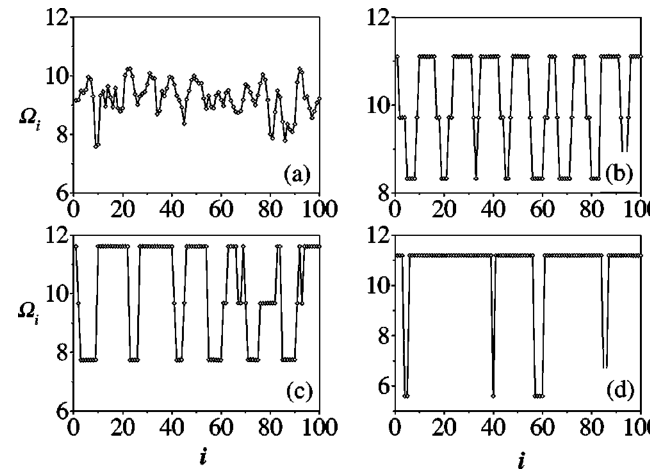

FIG. 5. The phase synchronization profile Ω_i versus i for $N = 100$ coupled nonidentical Lorenz oscillators with $K = 50$, $\Delta = 10.0$, and different bias couplings $\delta = 0, 30, 40, 50$ for (a), (b), (c), (d), respectively.

$$\begin{aligned} \dot{x}_i &= \sigma(y_i - x_i) + (K+\delta)(x_{i+1} - x_i) - (K-\delta)(x_i - x_{i-1}), \\ \dot{y}_i &= r_i x_i - y_i - x_i z_i, \\ \dot{z}_i &= -b z_i + x_i y_i. \end{aligned} \tag{7}$$

Here $\sigma = 10$, $b = 8/3$, and $r_i = 40 + \Delta_i$, $\Delta_i \in [-10, 10]$. $K = 50$ [16]. Although the trajectory of each oscillator has double rotation centers, one can still define a well-behaved phase for each oscillator due to the inversion symmetry $(x_i, y_i) \rightarrow (-x_i, -y_i)$:

$$\theta_i(t) = \tan^{-1}\left\{ \frac{[\sqrt{x_i^2 + y_i^2} - \sqrt{b(r_i - 1)}]}{[z_i - (r_i - 1)]} \right\}. \tag{8}$$

Here $(r_i - 1, \sqrt{b(r_i - 1)})$ is the rotation center of the ith oscillator. The average winding number can also be defined as the Rössler case. In Fig. 5, the profiles of Ω_i versus i for $\delta = 0, 30, 40$, and 50 are shown. The number of oscillators is $N = 100$. For $\delta = 0$, it can be seen from Fig. 5(a) that almost all oscillators are non-phase-synchronized due to the strong stochasticity of the Lorenz oscillator, and one is unable to observe even a small synchronized cluster. As $\delta = 30$, three nonlocal clusters are formed [see Fig. 5(b)]. As many nonidentical Lorenz oscillators are coupled to each other, full PS can hardly be achieved. When $\delta = 40$, the middle cluster becomes smaller. For $\delta=50$, one finds two clusters in Fig. 5(d), with a dominant cluster and a small one, i.e., more and more oscillators are phase locked to the larger cluster. Nonlocal PS can bring small clusters near the large cluster into a larger one. This enhancement of PS is different from the previous case for Rössler oscillators because Lorenz systems exhibit stronger stochasticity of chaos.

To summarize, in this paper we present enhancement of PS by asymmetric coupling in lattices of coupled chaotic Rössler and Lorenz oscillators. The mechanism responsible for this effect is the delocalization of the propagation of synchronized waves and nonlocal PS. The enhancement of PS via asymmetric couplings is a general behavior, which can be found in other chaotic systems. This effect should also be useful in applications, for in many cases synchronization is of great importance.

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